ELLIPSE [JEE ADVANCED PREVIOUS YEAR SOLVED PAPER]

JEE ADVANCED

Single Correct Answer Type

1. Let E be the ellipse $\frac{x^2}{9} + \frac{y^2}{4} = 1$ and C be the circle $x^2 + y^2 = 9$. Let P and Q be the points $(1, 2)$ and $(2, 1)$, respectively. Then,
   a. Q lies inside C but outside E
   b. Q lies outside both C and E
   c. P lies inside both C and E
   d. P lies inside C but outside E **(IIT-JEE 1994)**

2. The radius of the circle passing through the foci of the ellipse $\frac{x^2}{16} + \frac{y^2}{9}$ and having its center $(0, 3)$ is
   a. 4 b. 3 c. $\sqrt{12}$ d. 7/2
   (IIT-JEE 1995)

3. The number of values of c such that the straight line $y = 4x + c$ touches the curve $\frac{x^2}{4} + \frac{y^2}{1} = 1$ is
   a. 0 b. 1 c. 2 d. infinite
   (IIT-JEE 1998)

4. If $P = (x, y)$, $F_1 = (3, 0)$, $F_2 = (-3, 0)$, and $16x^2 + 25y^2 = 400$, then $PF_1 + PF_2$ equals
   a. 8 b. 6 c. 10 d. 12
   (IIT-JEE 1998)

5. The area of the quadrilateral formed by the tangents at the endpoint of the latus rectum to the ellipse $\frac{x^2}{9} + \frac{y^2}{5} = 1$ is
   a. 27/4 sq. units b. 9 sq. units
   c. 27/2 sq. units d. 27 sq. units
   (IIT-JEE 2003)

6. If tangents are drawn to the ellipse $x^2 + 2y^2 = 2$, then the locus of the midpoint of the intercept made by the tangents between the coordinate axes is
   a. $\frac{1}{2x^2} + \frac{1}{4y^2} = 1$ b. $\frac{1}{4x^2} + \frac{1}{2y^2} = 1$
   c. $\frac{x^2}{2} + \frac{y^2}{4} = 1$ d. $\frac{x^2}{4} + \frac{y^2}{2} = 1$
   (IIT-JEE 2004)

7. The minimum area of the triangle formed by the tangent to $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ and the coordinate axes is
   a. ab sq. units b. $\frac{a^2 + b^2}{2}$ sq. units
   c. $\frac{(a+b)^2}{2}$ sq. units d. $\frac{a^2 + ab + b^2}{3}$ sq. units
   (IIT-JEE 2004)

8. The line passing through the extremity A of the major axis and extremity B of the minor axis of the ellipse $x^2 + 9y^2 = 9$ meets its auxiliary circle at the point M. Then the area of the triangle with vertices at A, M, and O (the origin) is
   a. 31/10 b. 29/10 c. 21/10 d. 27/10
   (IIT-JEE 2009)

9. The normal at a point P on the ellipse $x^2 + 4y^2 = 16$ meets the x-axis at Q. If M is the midpoint of the line segment PQ, then the locus of M intersects the latus rectums of the given ellipse at points
   a. $(\pm(3\sqrt{5})/2, \pm 2/7)$
   b. $(\pm(3\sqrt{5})/2, \pm\sqrt{19}/7)$
   c. $(\pm 2\sqrt{3}, \pm 1/7)$
   d. $(\pm 2\sqrt{3}, \pm 4\sqrt{3}/7)$ **(IIT-JEE 2009)**

10. The ellipse E_1: $\frac{x^2}{9} + \frac{y^2}{4} = 1$ is inscribed in a rectangle R whose sides are parallel to the coordinate axes. Another ellipse E_2 passing through the point $(0, 4)$ circumscribes the rectangle R. The eccentricity of the ellipse E_2 is
    a. $\sqrt{2}/2$ b. $\sqrt{3}/2$ c. 1/2 d. 3/4
    (IIT-JEE 2012)

Multiple Correct Answers Type

1. On the ellipse $4x^2 + 9y^2 = 1$, the points at which the tangents are parallel to the line $8x = 9y$ are
   a. (2/5, 1/5) b. (– 2/5, 1/5)
   c. (– 2/5, –1/5) d. (2/5, –1/5)
   (IIT-JEE 1999)

2. Let $P(x_1, y_1)$ and $Q(x_2, y_2)$, $y_1 < 0$, $y_2 < 0$, be the endpoints of the latus rectum of the ellipse $x^2 + 4y^2 = 4$. The equations of parabolas with latus rectum PQ are
   a. $x^2 + 2\sqrt{3}y = 3 + \sqrt{3}$ b. $x^2 - 2\sqrt{3}y = 3 + \sqrt{3}$
   c. $x^2 + 2\sqrt{3}y = 3 - \sqrt{3}$ d. $x^2 - 2\sqrt{3}y = 3 - \sqrt{3}$
   (IIT-JEE 2008)

3. In a triangle ABC with fixed base BC, the vertex A moves such that $\cos B + \cos C = 4\sin^2\frac{A}{2}$. If a, b, and c denote the lengths of the sides of the triangle opposite to the angles A, B, and C, respectively, then
   a. $b + c = 4a$
   b. $b + c = 2a$
   c. the locus of point A is an ellipse
   d. the locus of point A is a pair of straight lines
   (IIT-JEE 2009)

4. Let E_1 and E_2 be two ellipses whose centers are at the origin. The major axes of E_1 and E_2 lie along the x-axis and the y-axis, respectively. Let S be the circle $x^2 + (y - 1)^2 = 2$. The straight line $x + y = 3$ touches the curves S, E_1 and E_2 at P, Q and R, respectively. Suppose that PQ

$= PR = \frac{2\sqrt{2}}{3}$. If e_1 and e_2 are the eccentricities of E_1 and E_2, respectively, then the correct expression(s) is (are)

a. $e_1^2 + e_2^2 = \frac{43}{40}$ **b.** $e_1 e_2 = \frac{\sqrt{7}}{2\sqrt{10}}$

c. $|e_1^2 - e_2^2| = \frac{5}{8}$ **d.** $e_1 e_2 = \frac{\sqrt{3}}{4}$

(JEE Advanced 2015)

Linked Comprehension Type

For Problems 1-3

Tangents are drawn from the point $P(3, 4)$ to the ellipse $\frac{x^2}{9} + \frac{y^2}{4} = 1$ touching the ellipse at points A and B.

(IIT-JEE 2010)

1. The coordinates of A and B are, respectively,
   a. (3, 0) and (0, 2)
   b. $(-8/5, 2\sqrt{161}/15)$ and $(-9/5, 8/5)$
   c. $(-8/5, 2\sqrt{161}/15)$ and (0, 2)
   d. (3, 0) and $(-9/5, 8/5)$

2. The orthocenter of triangle PAB is
   a. (5, 8/7) **b.** (7/5, 25/8)
   c. (11/5, 8/5) **d.** (8/25, 7/5)

3. The equation of the locus of the point whose distances from the point P and the line AB are equal is
   a. $9x^2 + y^2 - 6xy - 54x - 62y + 241 = 0$
   b. $x^2 + 9y^2 + 6xy - 54x + 62y - 241 = 0$
   c. $9x^2 + 9y^2 - 6xy - 54x - 62y - 241 = 0$
   d. $x^2 + y^2 - 2xy + 27x + 31y - 120 = 0$

Matching Column Type

1. Match the conics in Column I with the statements/ expressions in Column II.

Column I	Column II
(a) Circle	(p) The locus of the point (h, k) for which the line $hx + ky = 1$ touches the circle $x^2 + y^2 = 4$
(b) Parabola	(q) Points z in the complex plane satisfying $\lvert z+2\rvert - \lvert z-2\rvert = \pm 3$
(c) Ellipse	(r) Points of the conic have parametric representation $x = \sqrt{3}\left(\frac{1-t^2}{1+t^2}\right)$, $y = \frac{2t}{1+t^2}$
(d) Hyperbola	(s) The eccentricity of the conic lies in the interval $1 \le x < \infty$
	(t) Points z in the complex plane satisfying $\text{Re}(z+1)^2 = \lvert z\rvert^2 + 1$

(IIT-JEE 2009)

2. Match the following:

List–I	List–II
(p) Let $y(x) = \cos(3\cos^{-1}x)$, $x \in [-1, 1], x \neq \pm\frac{\sqrt{3}}{2}$. Then $\frac{1}{y(x)}\left\{(x^2-1)\frac{d^2y(x)}{dx^2} + x\frac{dy(x)}{dx}\right\}$ equals	(1) 1
(q) Let $A_1, A_2, \ldots, A_n$ $(n > 2)$ be the vertices of a regular polygon of n sides with its centre at the origin. Let $\vec{a}_k$ be the position vector of the point A_k, $k = 1, 2, \ldots n$. If $\left\lvert\sum_{k=1}^{n-1}(\vec{a}_k \times \vec{a}_{k+1})\right\rvert = \left\lvert\sum_{k=1}^{n-1}(\vec{a}_k \cdot \vec{a}_{k+1})\right\rvert$, then the minimum value of n is	(2) 2
(r) If the normal from the point $P(h, 1)$ on the ellipse $\frac{x^2}{6} + \frac{y^2}{3} = 1$ is perpendicular to the line $x + y = 8$, then the value of h is	(3) 8
(s) Number of positive solutions satisfying the equation $\tan^{-1}\left(\frac{1}{2x+1}\right) + \tan^{-1}\left(\frac{1}{4x+1}\right) = \tan^{-1}\left(\frac{2}{x^2}\right)$ is	(4) 9

Codes:

	(p)	(q)	(r)	(s)
a.	(4)	(3)	(2)	(1)
b.	(2)	(4)	(3)	(1)
c.	(4)	(3)	(1)	(2)
d.	(2)	(4)	(1)	(3)

(JEE Advanced 2014)

Integer Answer Type

1. A vertical line passing through the point $(h, 0)$ intersects the ellipse $\frac{x^2}{4} + \frac{y^2}{3} = 1$ at points P and Q. Let the tangents to the ellipse at P and Q meet at point R. If $\Delta(h)$ = area of triangle PQR, $\Delta_1 = \max_{1/2 \le h \le 1} \Delta(h)$, and $\Delta_2 = \min_{1/2 \le h \le 1} \Delta(h)$, then $\frac{8}{\sqrt{5}}\Delta_1 - 8\Delta_2 =$

(JEE Advanced 2013)

2. Suppose that the foci of the ellipse $\frac{x^2}{9}+\frac{y^2}{5}=1$ are $(f_1, 0)$ and $(f_2, 0)$ where $f_1 > 0$ and $f_2 < 0$. Let P_1 and P_2 be two parabolas with a common vertex at (0, 0) and with foci at $(f_1, 0)$ and $(2f_2, 0)$, respectively. Let T_1 be a tangent to P_1 which passes through $(2f_2, 0)$ and T_2 be a tangent to P_2 which passes through $(f_1, 0)$. If m_1 is the slope of T_1 and m_2 is the slope of T_2, then the value of $\left(\frac{1}{m_1^2}+m_2^2\right)$ is **(JEE Advanced 2015)**

Fill in the Blanks Type

1. An ellipse has OB as the semi-minor axis; F, F' as its foci; and $\angle FBF'$ is a right angle. Then, the eccentricity of the ellipse is ________. **(IIT-JEE 1997)**

Subjective Type

1. Let d be the perpendicular distance from the center of the ellipse to any tangent to the ellipse. If F_1 and F_2 are the two foci of the ellipse, then show that $(PF_1 - PF_2)^2 = 4a^2\left(1-\frac{b^2}{d^2}\right)$. **(IIT-JEE 1995)**
2. A tangent to the ellipse $x^2 + 4y^2 = 4$ meets the ellipse $x^2 + 2y^2 = 6$ at P and Q. Prove that the tangents at P and Q of the ellipse $x^2 + 2y^2 = 6$ are at right angles. **(IIT-JEE 1997)**
3. Consider the family of circles $x^2 + y^2 = r^2$, $2 < r < 5$. If in the first quadrant, the common tangent to a circle of this family and the ellipse $4x^2 + 25y^2 = 100$ meets the coordiante axes at A and B, then find the equation of the locus of the midpoint of AB. **(IIT-JEE 1999)**
4. Find the coordinates of all the points P on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ for which the area of triangle PON is maximum, where O denotes the origin and N is the foot of the perpendicular from O to the tangent at P. **(IIT-JEE 1999)**
5. Let ABC be an equilateral triangle inscribed in the circle $x^2 + y^2 = a^2$. Suppose perpendiculars from A, B, and C to the major axis of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, $(a > b)$, meet the ellipse, respectively, at P, Q, and R so that P, Q, and R lie on the same side of the major axis as A, B, and C, respectively. Prove that the normals to the ellipse drawn at the points P, Q, and R are concurrent. **(IIT-JEE 2000)**
6. Let P be a point on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, $0 < b < a$. Let the line parallel to the y-axis passing through P meets the circle $x^2 + y^2 = a^2$ at point Q such that P and Q are on the same side of the x-axis. For two positive real numbers r and s, find the locus of the point R on PQ such that $PR : RQ = r : s$ as P varies over the ellipse. **(IIT-JEE 2001)**
7. Prove that in an ellipse, the perpendicular from a focus upon any tangent and the line joining the center of the ellipse to the point of contact meet at the corresponding directrix. **(IIT-JEE 2002)**
8. From a point, common tangents are drawn to the curves $x^2 + y^2 = 16$ and $\frac{x^2}{25}+\frac{y^2}{4}=1$. Find the slope of the common tangent in the first quadrant and also find the length of the intercept between the coordinate axes. **(IIT-JEE 2005)**

Answer Type

JEE Advanced

Single Correct Answer Type

1. d. **2.** a. **3.** c. **4.** c. **5.** d.
6. a. **7.** a. **8.** d. **9.** c. **10.** c.

Multiple Correct Answers Type

1. b., d. **2.** b., c. **3.** b., c. **4.** a., b

Linked Comprehension Type

1. d. **2.** c. **3.** a.

Matching Column Type

1. (r) – (c) **2.** a.

Integer Answer Type

1. 9 **2.** 4

Fill in the Blanks Type

1. $\frac{1}{\sqrt{2}}$

Subjective Type

3. $\frac{25}{x^2}+\frac{4}{y^2}=4$ **4.** $\left(\pm\frac{a^2}{\sqrt{a^2+b^2}}, \pm\frac{b^2}{\sqrt{a^2+b^2}}\right)$

6. $\frac{x^2}{a^2}+\frac{y^2(r+s)^2}{(bs+ar)^2}=1$ **8.** $m=-\frac{2}{\sqrt{3}}, \frac{14}{\sqrt{3}}$

Hints and Solutions

JEE Advanced

Single Correct Answer Type

1. **d.** Since $1^2 + 2^2 = 5 < 9$ and $2^2 + 1^2 = 5 < 9$, both P and Q lie inside C. Also,

$$\frac{1^2}{9} + \frac{2^2}{4} = \frac{1}{9} + 1 > 1 \text{ and } \frac{2^2}{9} + \frac{1}{4} = \frac{25}{36} < 1$$

Hence, P lies outside E and Q lies inside E. Thus, P lies inside C but outside E.

2. **a.** The given ellipse is

$$\frac{x^2}{16} + \frac{y^2}{9} = 1$$

Here, $a^2 = 16$ and $b^2 = 9$

$\therefore b^2 = a^2(1 - e^2)$

or $9 = 16(1 - e^2)$

or $e = \dfrac{\sqrt{7}}{4}$

Hence, the foci are $(\pm\sqrt{7}, 0)$.

Radius of circle = Distance between $(\pm\sqrt{7}, 0)$ and $(0, 3)$

$$= \sqrt{7 + 9} = 4$$

3. **c.** For given slope, there exist two parallel tangents to the ellipse. Hence, there are two values of c.

4. **c.** The ellipse can be written as

$$\frac{x^2}{25} + \frac{y^2}{16} = 1$$

Here, $a^2 = 25, b^2 = 16$.

Now, $b^2 = a^2(1 - e^2)$

or $\frac{16}{25} = 1 - e^2$

or $e^2 = 1 - \frac{16}{25} = \frac{9}{25}$

or $e = \frac{3}{5}$

The foci of the ellipse are $(\pm ae, 0) \equiv (\pm 3, 0)$, i.e., F_1 and F_2 are the foci of the ellipse.

Therefore, we have $PF_1 + PF_2 = 2a = 10$ for every point P on the ellipse.

5. d. The given ellipse is

$$\frac{x^2}{9} + \frac{y^2}{5} = 1$$

Then, $a^2 = 9, b^2 = 5$. Therefore,

$$e = \sqrt{1 - \frac{5}{9}} = \frac{2}{3}$$

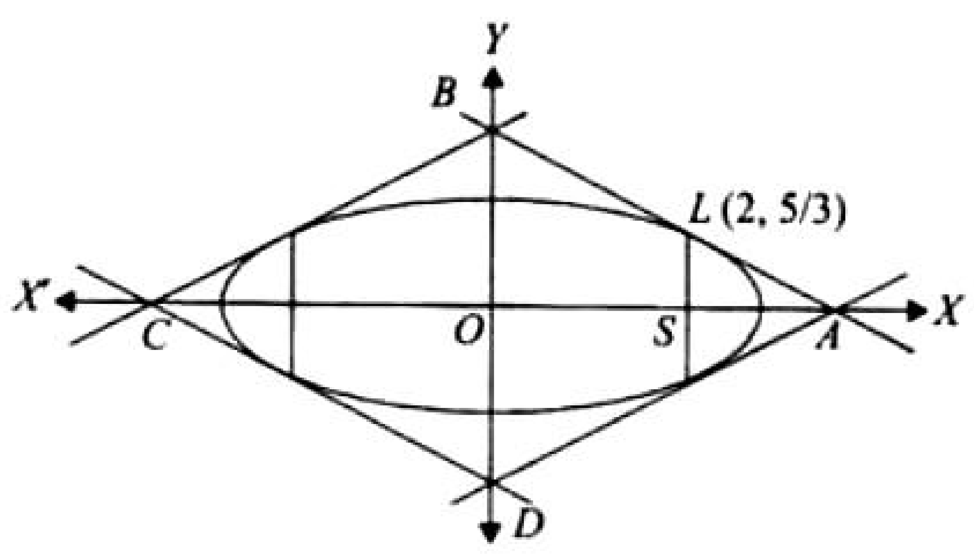
Hence, the endpoint of latus rectum in first quadrant is $L(2, 5/3)$.

The equation of tangent at L is

$$\frac{2x}{9} + \frac{y}{3} = 1$$

The tangent meets the x-axis at $A(9/2, 0)$ and the y-axis at $B(0, 3)$. Therefore,

$$\text{Area of } \Delta OAB = \frac{1}{2} \times \frac{9}{2} \times 3 = \frac{27}{4}$$



By symmetry,

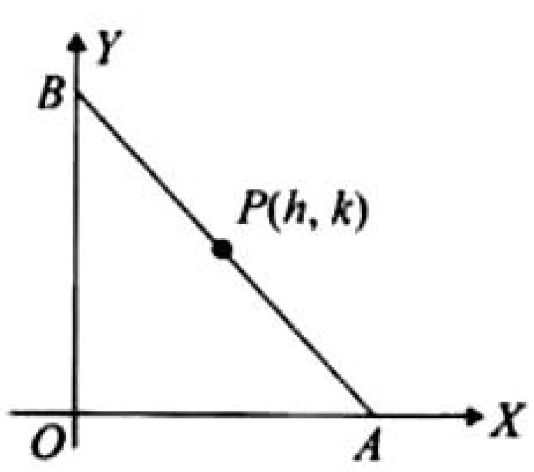
Area of quadrilateral $= 4 \times (\text{Area of } \Delta OAB)$

$= 4 \times \frac{27}{4} = 27$ sq. units

6. a. Any tangent to ellipse

$\frac{x^2}{2} + \frac{y^2}{1} = 1$ is given by

$$\frac{x \cos\theta}{\sqrt{2}} + y \sin\theta = 1$$



Let it meet axes at A and B.

$\therefore \quad A \equiv (\sqrt{2} \sec\theta, 0)$ and $B \equiv (0, \operatorname{cosec}\theta)$.

Let $p(h, k)$ be the mid point of AB.

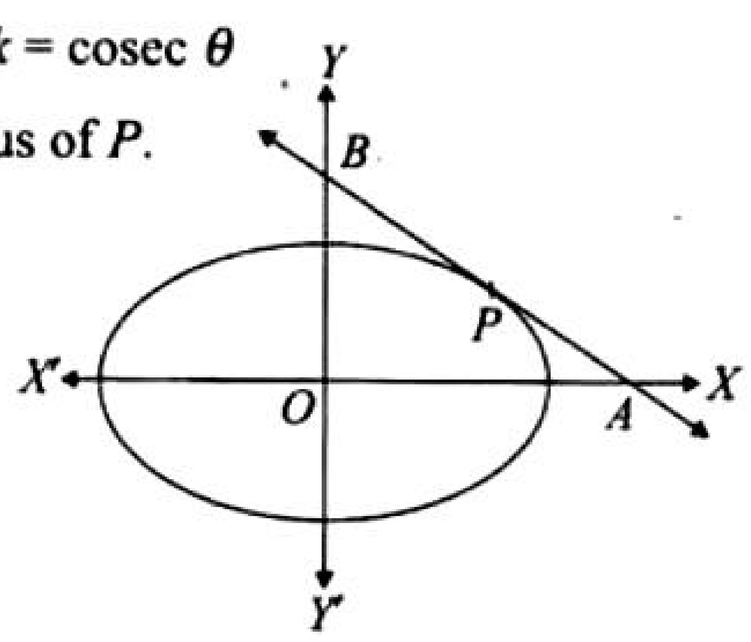
Hence, $2h = \sqrt{2} \sec\theta$ and $2k = \operatorname{cosec}\theta$

Eliminting 'θ', which is locus of P.

$$\left(\frac{1}{\sqrt{2}h}\right)^2 + \left(\frac{1}{2k}\right)^2 = 1$$

or $\frac{1}{2h^2} + \frac{1}{4k^2} = 1$

or $\frac{1}{2x^2} + \frac{1}{4y^2} = 1$



7. a. Tangent to the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

at $P(a \cos\theta, b \sin\theta)$ is given by

$$\frac{x \cos\theta}{a} + \frac{y \sin\theta}{b} = 1$$

It meets the coordinate axes at $A(a \sec\theta, 0)$ and $B(0, b \operatorname{cosec}\theta)$. Therefore,

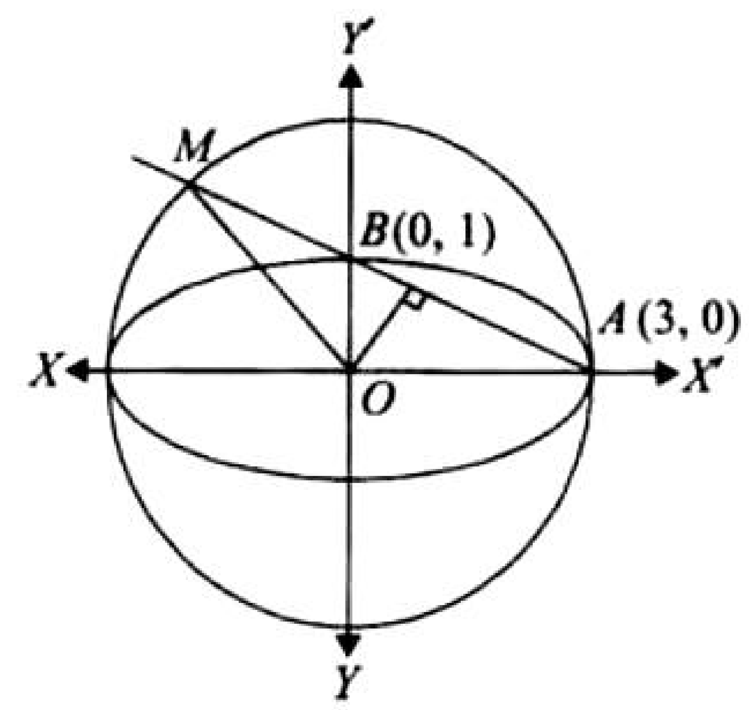
$$\text{Area of } \Delta OAB = \frac{1}{2} \times a \sec\theta \times b \operatorname{cosec}\theta$$

or $\Delta = \frac{ab}{\sin 2\theta}$

For area to be minimum, $\sin 2\theta$ should be maximum and we know that the maximum value of $\sin 2\theta$ is 1.Therefore,

$\Delta_{max} = ab$

8. d.



The equation of line AM is

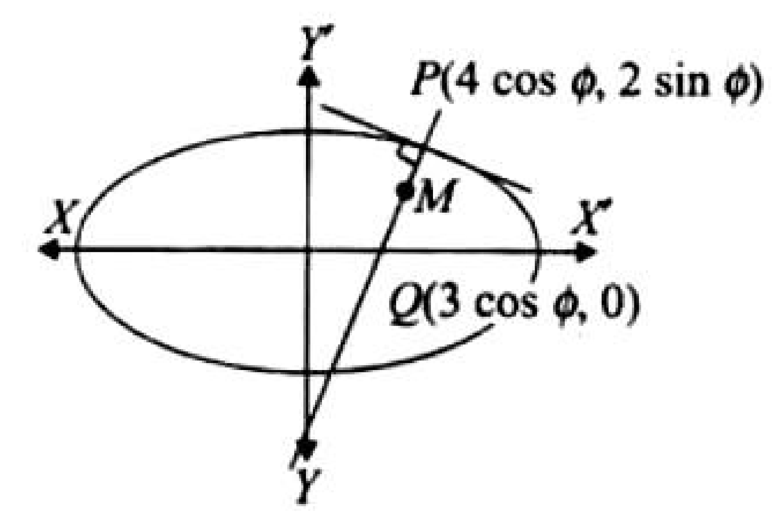
$x + 3y - 3 = 0$

Perpendicular distance of line from the origin $= \frac{3}{\sqrt{10}}$

Length of $AM = 2\sqrt{9 - \frac{9}{10}} = 2 \times \frac{9}{\sqrt{10}}$

$\therefore$ Area of $\Delta OAM = \frac{1}{2} \times 2 \times \frac{9}{\sqrt{10}} \times \frac{3}{\sqrt{10}} = \frac{27}{10}$ sq. units.

9. c. Given ellipse is $\frac{x^2}{16} + \frac{y^2}{4} = 1$. Normal at $P(4 \cos\phi, 2 \sin\phi)$ is given by $4x \sec\phi - 2y \operatorname{cosec}\phi = 12$



It meets x-axis at

$Q \equiv (3 \cos\phi, 0)$

Let $M \equiv (\alpha, \beta)$ be the mid point of PQ.

$\therefore \alpha = \frac{3 \cos\phi + 4 \cos\phi}{2} = \frac{7}{2} \cos\phi$

or $\cos\phi = \frac{2}{7}\alpha$

and $\beta = \sin\phi$

Using $\cos^2\phi + \sin^2\phi = 1$, we have

$$\frac{4}{49}\alpha^2 + \beta^2 = 1$$

or $\frac{4}{49}x^2 + y^2 = 1$ (i)

Now, the latus rectum is

$x = \pm 2\sqrt{3}$ (ii)

Solving (i) and (ii), we have

$$\frac{48}{49} + y^2 = 1$$

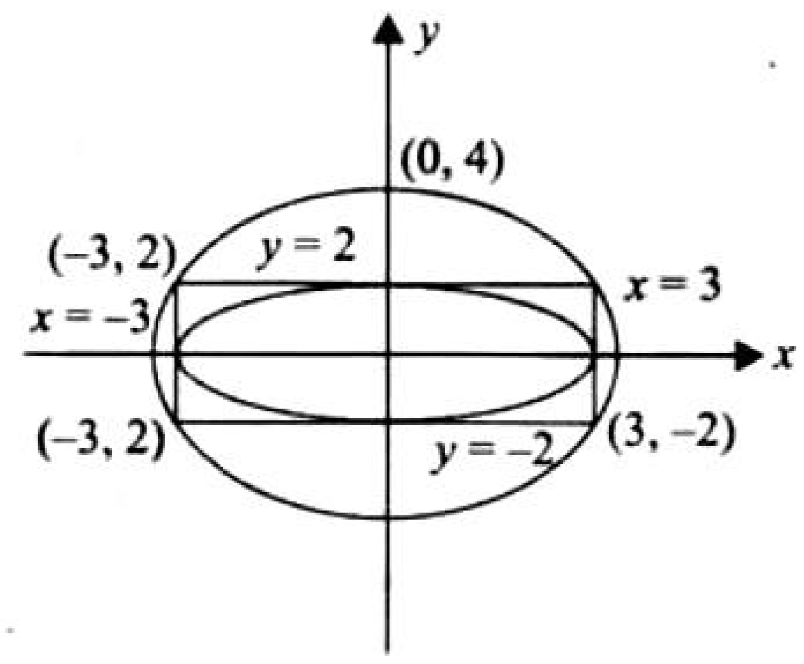
or $y = \pm\frac{1}{7}$

The points of intersection are $(\pm 2\sqrt{3}, \pm 1/7)$.

10. c. Let the ellipse be

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

as it is passing through (0, 4) and (3, 2). So,



$b^2 = 16$ and $\frac{9}{a^2} + \frac{4}{16} = 1$

or $a^2 = 12$

So, $12 = 16(1 - e^2)$

or $e = \frac{1}{2}$.

Multiple Correct Answers Type

1. b., d. Let (x_1, y_1) be the point at which tangents to the ellipse $4x^2 + 9y^2 = 1$ are parallel to $8x = 9y$.

Then the slope of the tangent is 8/9, i.e.,

$\left(\frac{dy}{dx}\right)_{(x_1, y_1)} = \frac{8}{9}$ (i)

Differentiating the equation of ellipse w.r.t. x, we get

$$8x + 18y\frac{dx}{dx} = 0$$

or $\left(\frac{dy}{dx}\right)_{(x_1, y_1)} = \frac{-8x_1}{18y_1} = \frac{-4x_1}{9y_1}$

Substituting in (i), we get

$$\frac{-4x_1}{9y_1} = \frac{8}{9}$$

or $-x_1 = 2y_1$ (ii)

Also, (x_1, y_1) is the point of contact which must be on the curve.

Hence,

$$4x_1^2 + 9y_1^2 = 1$$

or $4 \times 4y_1^2 + 9y_1^2 = 1$ [Using (ii)]

or $y_1^2 = \frac{1}{25}$

or $y_1 = \pm\frac{1}{5}$

$\therefore x_1 = \mp\frac{2}{5}$

Thus, the required points are (–2/5, 1/5) and (2/5, –1/5).

Alternative Method:

Let $y = \frac{8}{9}x + c$

be the tangent to

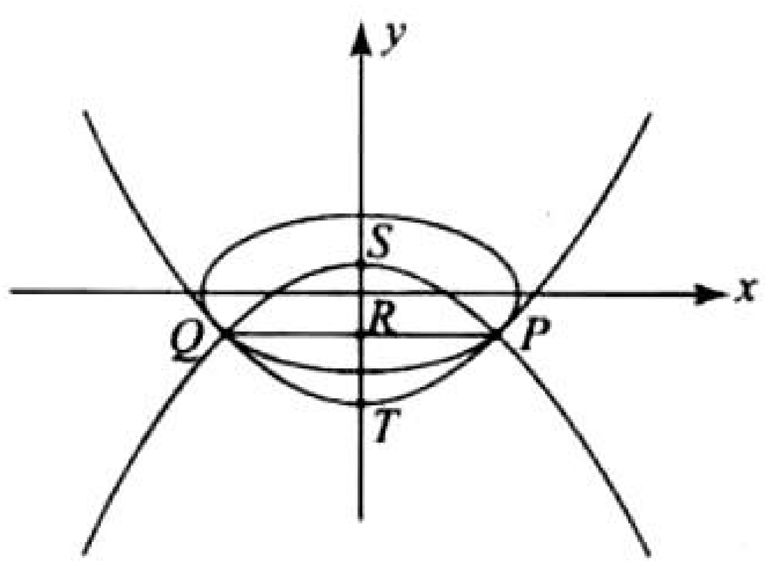
$$\frac{x^2}{1/4} + \frac{y^2}{1/9} = 1$$

where $c = \pm\sqrt{a^2m^2 + b^2} = \pm\sqrt{\frac{1}{4} \times \frac{64}{81} + \frac{1}{9}} = \pm\frac{5}{9}$

So, the points of contact are $(-a^2m/c, b^2/c) \equiv (2/5, -1/5)$ or $(-2/5, 1/5)$.

2. b., c. The given ellipse is

$$\frac{x^2}{4} + \frac{y^2}{1} = 1$$



$b^2 = a^2(1 - e^2)$

or $e = \frac{\sqrt{3}}{2}$

Hence, the endpoints P and Q of the latus rectum are given by

$P \equiv \left(\sqrt{3}, -\frac{1}{2}\right)$

and $Q \equiv \left(-\sqrt{3}, -\frac{1}{2}\right)$ (Given $y_1, y_2 < 0$)

The coordinates of the midpoint of PQ are

$R \equiv \left(0, -\frac{1}{2}\right)$

Length of latus rectum, $PQ = 2\sqrt{3}$

Hence, two parabolas are possible whose vertices are

$S\left(0, +\frac{\sqrt{3}}{2} - \frac{1}{2}\right)$ and $T\left(0, -\frac{\sqrt{3}}{2} - \frac{1}{2}\right)$

The equations of the parabolas are

$$x^2 = 2\sqrt{3}\left(y + \frac{\sqrt{3}}{2} + \frac{1}{2}\right)$$

and $x^2 = -2\sqrt{3}\left(y - \frac{\sqrt{3}}{2} + \frac{1}{2}\right)$

or $x^2 - 2\sqrt{3}y = 3 + \sqrt{3}$

and $x^2 + 2\sqrt{3}y = 3 - \sqrt{3}$

3. b., c.

Given $\cos B + \cos C = 4\sin^2\frac{A}{2}$

or $2\cos\left(\frac{B+C}{2}\right)\cos\left(\frac{B-C}{2}\right) = 4\sin^2\frac{A}{2}$

or $\cos\left(\frac{B-C}{2}\right) = 2\sin\left(\frac{A}{2}\right)$

or $2\sin\frac{B+C}{2}\cos\frac{B-C}{2} = 4\sin\frac{A}{2}\cos\frac{A}{2}$

or $\sin B + \sin C = 2 \sin A$
(using sine rule)
Thus sum of two variable sides is constant.
or $b + c = 2a$ (constant)
Hence, the locus of vertex A is an ellipse with B and C as foci.

4. a., b.

Let ellipses be $E_1 : \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ and $E_2 = \frac{x^2}{A^2} + \frac{y^2}{B^2} = 1$
Since $x + y = 3$ is a tangent,
$a^2 + b^2 = A^2 + B^2 = 9$ (using condition $c^2 = a^2m^2 + b^2$ etc.)
Point P lies on $x^2 + (y - 1)^2 = 2$.
Equation of normal to circle having slope 1 is $y - 1 = 1(x - 0)$
or $x - y + 1 = 0$
Solving this normal with tangent line we get point $P(1, 2)$.

Now $PQ = PR = \frac{2\sqrt{2}}{3}$

So, points on line $x + y - 3 = 0$ at distance $\frac{2\sqrt{2}}{3}$ from point P

are $\left(1 \mp \frac{1}{\sqrt{2}} \frac{2\sqrt{2}}{3}, 2 \mp \frac{1}{\sqrt{2}} \frac{2\sqrt{2}}{3}\right)$

or $Q\left(\frac{5}{3}, \frac{4}{3}\right)$ and $Q\left(\frac{1}{3}, \frac{8}{3}\right)$

Now $Q\left(\frac{5}{3}, \frac{4}{3}\right)$ lies on E_1

So, $\frac{25}{9a^2} + \frac{16}{9(9 - a^2)} = 1$

$\Rightarrow \quad 225 - 25a^2 + 16a^2 = 9a^2(9 - a^2)$
$\Rightarrow \quad a^4 - 10a^2 + 25 = 0$
$\Rightarrow \quad a^2 = 5$ so $b^2 = 4$

$\therefore \quad e_1^2 = 1 - \frac{b^2}{a^2} = 1 - \frac{4}{5} = \frac{1}{5}$

Now $\left(\frac{1}{3}, \frac{8}{3}\right)$ lies on E_2

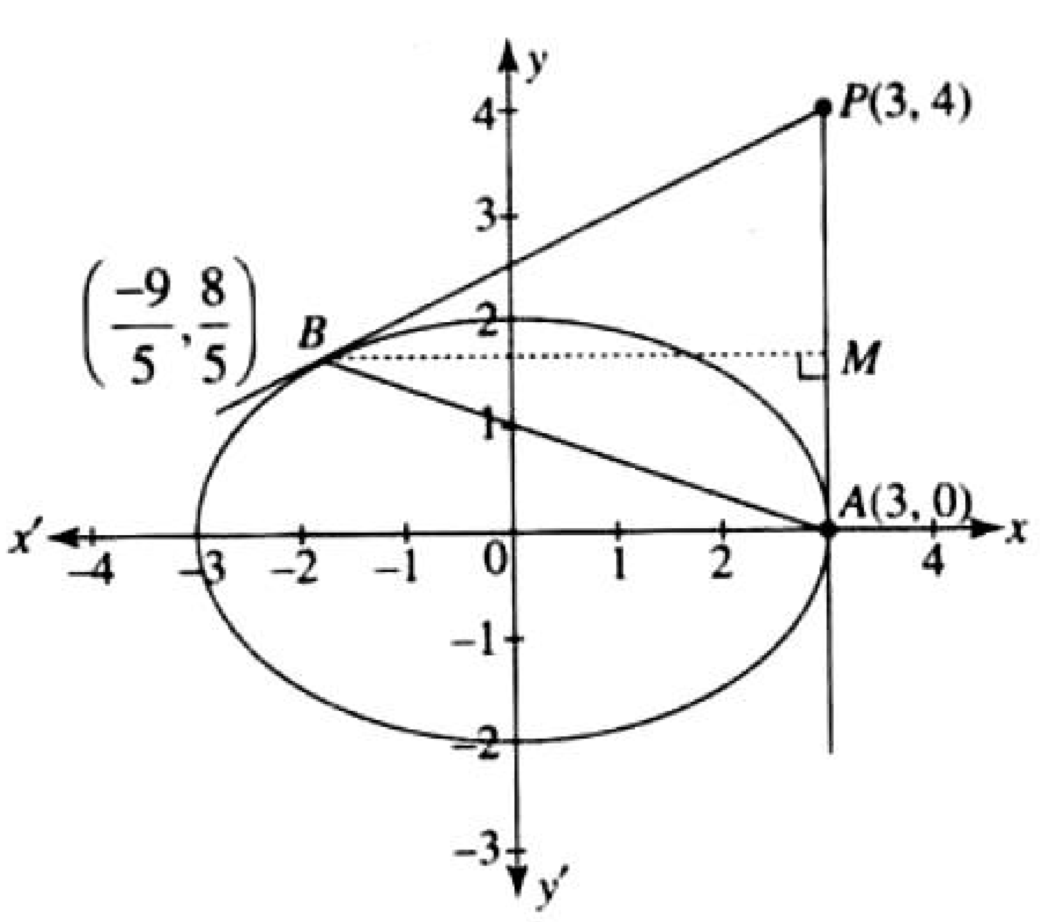
So, $\frac{1}{A^2} + \frac{64}{(9 - A^2)} = 9$

$\Rightarrow \quad 9 - A^2 + 64A^2 = 9A^2(9 - A^2)$
$\Rightarrow \quad A^4 - 2A^2 + 1 = 0$
$\Rightarrow \quad A^2 = 1 \qquad$ so, $B^2 = 8$

$\therefore \quad e_2^2 = 1 - \frac{A^2}{B^2} = 1 - \frac{1}{8} = \frac{7}{8}$

Linked Comprehension Type

1. d.



Given ellipse $\frac{x^2}{9} + \frac{y^2}{4} = 1$
The equation of tangent having slope m is

$$y = mx + \sqrt{9m^2 + 4}$$

The tangent passes through the point (3, 4). Therefore,

$$4 - 3m = \sqrt{9m^2 + 4}$$

Squaring, we have

$$16 + 9m^2 - 24m = 9m^2 + 4$$

or $m = \frac{12}{24} = \frac{1}{2}$

Therefore, the equation of tangent is

$$y - 4 = \frac{1}{2}(x - 3) \text{ or } x - 2y + 5 = 0 \qquad \text{(i)}$$

Let the point of contact on the curve be $B(\alpha, \beta)$.
Equation of tangent at this point is

$$\frac{x\alpha}{9} + \frac{y\beta}{4} - 1 = 0 \qquad \text{(ii)}$$

Comparing equations (i) and (ii)

or $\frac{\alpha/9}{1} = \frac{\beta/4}{-2} = -\frac{1}{5}$

or $\alpha = -\frac{9}{5}, \beta = \frac{8}{5}$

$B \equiv \left(-\frac{9}{5}, \frac{8}{5}\right)$

Another slope of tangent is ∞. Then the equation of tangent is $x = 3$ and the corresponding point of contact is $A(3, 0)$.

2. c. Since the slope of PA is ∞, the slope of altitude through B must be 0, for which the orthocenter is (11/5, 8/5).

3. a. The equation of chord of contact AB is

$\frac{3x}{9} + \frac{4y}{4} = 1$ or $\frac{x}{3} + y = 1$ or $x + 3y - 3 = 0$

Thus, equation of locus of the point whose distances from the point P and the line AB are equal is

$$(x - 3)^2 + (y - 4)^2 = \frac{(x + 3y - 3)^2}{10}$$

or $9x^2 + y^2 - 6xy - 54x - 62y + 241 = 0$

Matching Column Type

1. (r) – (c)

$$x = \sqrt{3}\left(\frac{1 - t^2}{1 + t^2}\right), y = \frac{2t}{1 + t^2}$$

Let $t = \tan \alpha$

$\therefore \cos 2\alpha = \frac{x}{\sqrt{3}}$ and $\sin 2\alpha = y$

$\therefore \frac{x^2}{3} + y^2 = \sin^2 2\alpha + \cos^2 2\alpha = 1$, which is an ellipse.

Note: Solutions of the remaining parts are given in their respective chapters.

2. a.

For statement (r):

Point $(h, 1)$ lies on ellipse $\frac{x^2}{6}+\frac{y^2}{3}=1$

$\Rightarrow \quad \frac{h^2}{6}+\frac{1^2}{3}=1$

$\therefore \quad h=\pm 2$

Tangent at $(2, 1)$ is $\frac{2x}{6}+\frac{y}{3}=1$

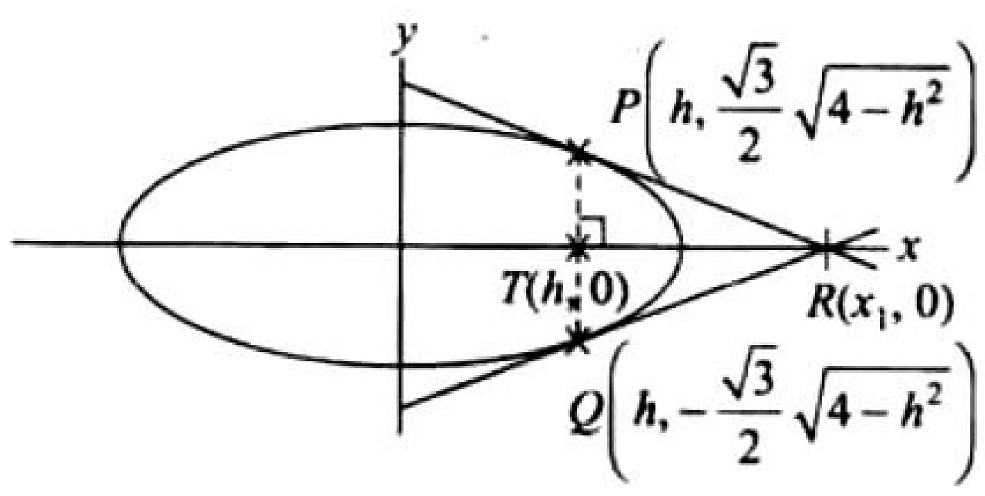
or $x+y=3$ which is parallel to $x+y=8$

$\therefore \quad h=2$

Note: Solutions of the remaining parts are given in their respective chapters.

Integer Answer Type

1. (9)



$\frac{x^2}{4}+\frac{y^2}{3}=1$

$\therefore y=\frac{\sqrt{3}}{2}\sqrt{4-h^2}$ at $x=h$

Let $R(x_1, 0)$.

PQ is the chord of contact. So,

$\frac{xx_1}{4}=1$

or $x=\frac{4}{x_1}$

which is the equation of PQ. At $x=h$,

$\frac{4}{x_1}=h$ or $x_1=\frac{4}{h}$

$\Delta(h)=$ Area of $\Delta PQR=\frac{1}{2}\times PQ\times RT$

$=\frac{1}{2}\times\frac{2\sqrt{3}}{2}\sqrt{4-h^2}\times(x_1-h)$

$=\frac{\sqrt{3}}{2h}(4-h^2)^{3/2}$

$\Delta'(h)=\frac{-\sqrt{3}(4+2h^2)}{2h^2}\sqrt{4-h^2}<0$

Therefore, $\Delta(h)$ is always decreasing. So,

$\Delta_1=$ Maximum of $\Delta(h)=\frac{45\sqrt{5}}{8}$ at $h=\frac{1}{2}$

and $\Delta_2=$ Minimum of $\Delta(h)=\frac{9}{2}$ at $h=1$

So, $\frac{8}{\sqrt{5}}\Delta_1-8\Delta_2=\frac{8}{\sqrt{5}}\times\frac{45\sqrt{5}}{8}-8\times\frac{9}{2}=45-36=9$

2.(4) Ellipse is $\frac{x^2}{9}+\frac{y^2}{5}=1$

$\therefore \quad a^2e^2=a^2-b^2=9-5=4$

$\therefore \quad$ foci are $(2, 0)$ and $(-2, 0)$

$\therefore \quad f_1=2$ and $f_2=-2$

Parabola with vertex $O(0, 0)$ and focus $(f_1, 0)$ or $(2, 0)$ is $y^2=8x$.

Equation of tangent having slope m_1 is

$y=m_1x+\frac{2}{m_1}$

It passes through $(2f_2, 0)$ or $(-4, 0)$.

$\therefore \quad 0=-4m_1+\frac{2}{m_1}$

$\Rightarrow \quad m_1=\pm\frac{1}{\sqrt{2}}$

Parabola with vertex $O(0, 0)$ and focus $(2f_2, 0)$ or $(-4, 0)$ is $y^2=-16x$.

Equation of tangent having slope m_2 is

$y=m_2x-\frac{4}{m_2}$

It passes through $(f_1, 0)$ or $(2, 0)$.

$\therefore \quad 0=2m_2-\frac{4}{m_2}$

$\Rightarrow \quad m_2=\pm\sqrt{2}$

$\Rightarrow \quad \frac{1}{m_1^2}+m_2^2=2+2=4$

Fill in the Blanks Type

1. Consider ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

Given $m_{BF}\cdot m_{BF'}=-1$

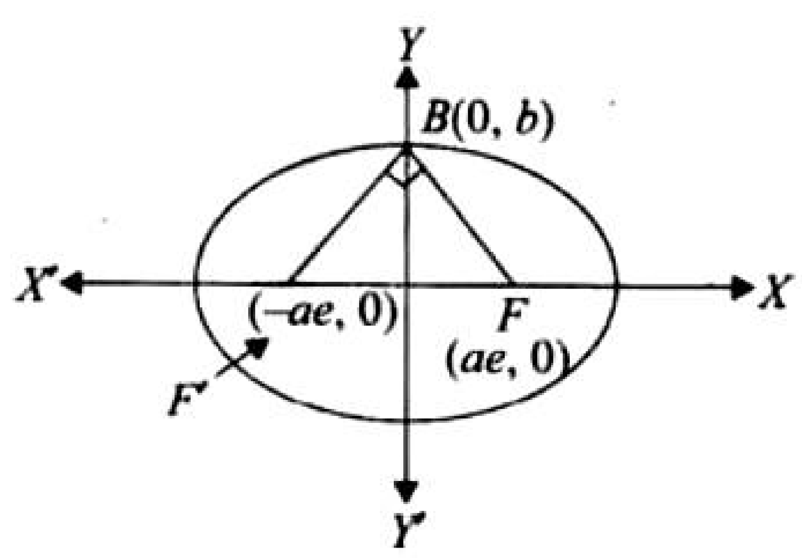
$\therefore \frac{b-0}{0-ae}\times\frac{b-0}{0+ae}=-1$

$\therefore \frac{b^2}{a^2e^2}=1$

$\therefore e^2=\frac{b^2}{a^2}=1-e^2$

$\therefore e^2=\frac{1}{2}$

$\therefore e=\frac{1}{\sqrt{2}}$



Subjective Type

1. The equation of the tangent at the point $P(a\cos\theta, b\sin\theta)$ on $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

is $\frac{x}{a}\cos\theta+\frac{y}{b}\sin\theta=1$ (i)

The perpendicular distance of (i) from the center $(0, 0)$ of the ellipse is given by

$d=\frac{1}{\sqrt{\frac{1}{a^2}\cos^2\theta+\frac{1}{b^2}\cos^2\theta}}=\frac{ab}{\sqrt{b^2\cos^2\theta+a^2\sin^2\theta}}$

$$\therefore\ 4a^2\left(1-\frac{b^2}{d^2}\right)=4a^2\left\{1-\frac{b^2\cos^2\theta+a^2\sin^2\theta}{a^2}\right\}$$

$$=4(a^2-b^2)\cos^2\theta=4a^2e^2\cos^2\theta \quad \text{(ii)}$$

The foci are $F_1 \equiv (ae, 0)$ and $F_2 \equiv (-ae, 0)$. Therefore,

$PF_1 = e(1 - e\cos\theta)$

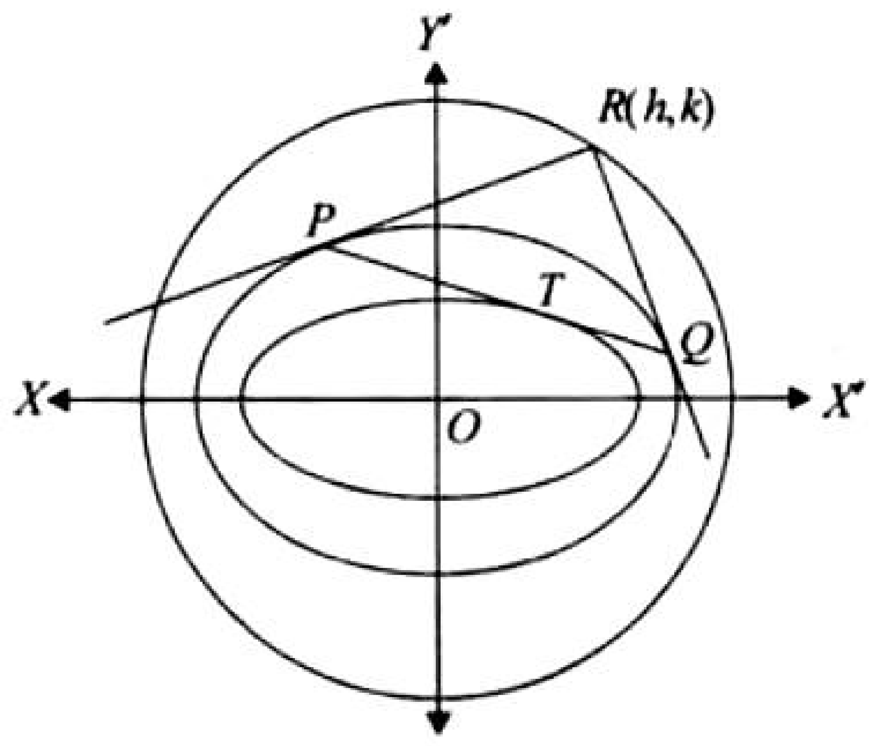
and $PF_2 = a(1 + e\cos\theta)$

$\therefore (PF_1 - PF_2)^2 = 4a^2e^2\cos^2\theta$ (iii)

Hence, from (ii) and (iii), we have

$$(PF_1-PF_2)^2=4a^2\left(1-\frac{b^2}{d^2}\right)$$

2.



The given ellipses are

$$\frac{x^2}{4}+\frac{y^2}{1}=1 \quad \text{(i)}$$

and $$\frac{x^2}{6}+\frac{y^2}{3}=1 \quad \text{(ii)}$$

Then the equation of tangent to (i) at any point $T(2\cos\theta, \sin\theta)$ is given by

$$\frac{x\cos\theta}{2}+\frac{y\sin\theta}{1}=1 \quad \text{(iii)}$$

Let this tangent meets the ellipse (ii) at P and Q. Let the tangents drawn to ellipse (ii) at P and Q meet each other at $R(h, k)$.

Then PQ is the chord of contact of ellipse (ii) with respect to the point $R(h, k)$ and is given by

$$\frac{xh}{6}+\frac{yk}{3}=1 \quad \text{(iv)}$$

Clearly, (iii) and (iv) represent the same line and, hence, should be identical.

Therefore, comparing the ratio of coefficients, we get

$$\frac{(\cos\theta)/2}{h/6}=\frac{\sin\theta}{k/3}=\frac{1}{1}$$

or $h = 3\cos\theta$, $k = 3\sin\theta$

or $h^2 + k^2 = 9$

Therefore, the locus of (h, k) is

$x^2 + y^2 = 9$

which is the director circle of the ellipse

$$\frac{x^2}{6}+\frac{y^2}{3}=1$$

and we know that the director circle is the locus of the point of intersection of the tangents which are at right angle.

Thus, tangents at P and Q are perpendicular.

3. Let the midpoint of AB be (h, k). Then the coordinates of A and B are $(2h, 0)$ and $(0, 2k)$, respectively.

Then the equation of line AB is

$$\frac{x}{2h}+\frac{y}{2k}=1 \text{ or } y=-\frac{k}{h}x+2k$$

It touches the ellipse

$$\frac{x^2}{25}+\frac{y^2}{4}=1$$

if $4k^2=25\left(-\frac{k}{h}\right)^2+4$

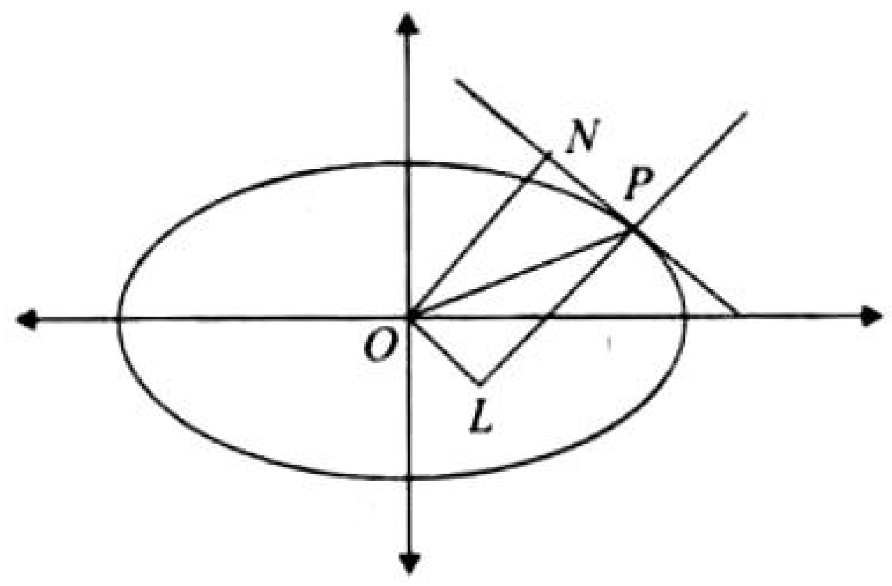
or $$\frac{25}{h^2}+\frac{4}{k^2}=4$$

Therefore, the locus of (h, k) is

$$\frac{25}{x^2}+\frac{4}{y^2}=4$$

(For the given tangent to the ellipse, the radius of the circle is automatically fixed.)

4.



The ellipse is

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

Since this ellipse is symmetrical in all four quadrants, either there exists no such point P or there are four points, one in each quadrant. Without loss of generality, we can assume that $a > b$ and P lies in the first quadrant.

Let P be $(a\cos\theta, b\sin\theta)$. Then the equation of tangent is

$$\frac{x}{a}\cos\theta+\frac{y}{b}\sin\theta=1$$

$$\therefore\ ON=\frac{ab}{\sqrt{b^2\cos^2\theta+a^2\sin^2\theta}}$$

The equation of ON is

$$\frac{x}{b}\sin\theta-\frac{y}{a}\cos\theta=0$$

The equation of normal at P is

$ax\sec\theta - by\,\text{cosec}\,\theta = a^2 - b^2$

$$\therefore\ OL=\frac{a^2-b^2}{\sqrt{a^2\sec^2\theta+b^2\,\text{cosec}^2\theta}}$$

$$=\frac{(a^2-b^2)\sin\theta\cos\theta}{\sqrt{a^2\sin^2\theta+b^2\cos^2\theta}}$$

Now, $NP = OL$

$$\therefore\ NP=\frac{(a^2-b^2)\sin\theta\cos\theta}{\sqrt{a^2\sin^2\theta+b^2\cos^2\theta}}$$

$\therefore \Delta$ = Area of triangle OPN

$$=\frac{1}{2}\times ON\times NP$$

$$= \frac{1}{2} ab(a^2 - b^2) \frac{\sin\theta\cos\theta}{a^2\sin^2\theta + b^2\cos^2\theta}$$

$$= \frac{1}{2} ab(a^2 - b^2) \frac{1}{a^2\tan\theta + b^2\cot\theta}$$

$$= \frac{1}{2} ab(a^2 - b^2) \frac{1}{(a\sqrt{\tan\theta} - b\sqrt{\cot\theta})^2 + 2ab}$$

Now, Δ is maximum when

$a\sqrt{\tan\theta} - b\sqrt{\cot\theta} = 0$ or $\tan\theta = \frac{b}{a}$

Therefore, P has coordinates $\left(a^2/\sqrt{a^2+b^2}, b^2/\sqrt{a^2+b^2}\right)$.

By symmetry, we have four such points, i.e.,

$$\left(\pm\frac{a^2}{\sqrt{a^2+b^2}}, \pm\frac{b^2}{\sqrt{a^2+b^2}}\right).$$

5. Let A, B, and C be the points on the circle whose coordinates are

$A(a\cos\theta, a\sin\theta)$

$B\left(a\cos\left(\theta + \frac{2\pi}{3}\right), a\sin\left(\theta + \frac{2\pi}{3}\right)\right)$

$C\left(a\cos\left(\theta + \frac{4\pi}{3}\right), a\sin\left(\theta + \frac{4\pi}{3}\right)\right)$

Hence, $P \equiv (a\cos\theta, b\sin\theta)$ [Given]

$Q \equiv \left(a\cos\left(\theta + \frac{2\pi}{3}\right), b\sin\left(\theta + \frac{2\pi}{3}\right)\right)$

$R \equiv \left(a\cos\left(\theta + \frac{4\pi}{3}\right), b\sin\left(\theta + \frac{4\pi}{3}\right)\right)$

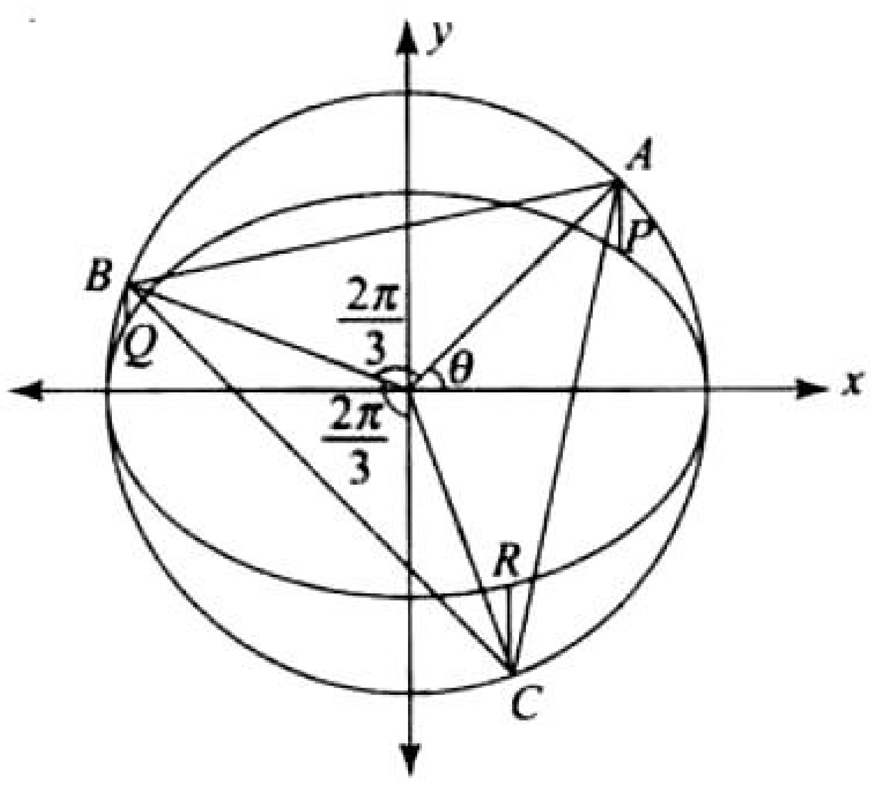
It is given that P, Q, and R are on the same side of the x-axis as A, B, and C.

So, the required normals to the ellipse at P, Q and R are

$ax\sec\theta - by\,\mathrm{cosec}\,\theta = a^2 - b^2$ (i)

$ax\sec\left(\theta + \frac{2\pi}{3}\right) - by\,\mathrm{cosec}\left(\theta + \frac{2\pi}{3}\right) = a^2 - b^2$ (ii)

$ax\sec\left(\theta + \frac{4\pi}{3}\right) - by\,\mathrm{cosec}\left(\theta + \frac{4\pi}{3}\right) = a^2 - b^2$ (iii)



Now, $\Delta = \begin{vmatrix} \sec\theta & \mathrm{cosec}\,\theta & 1 \\ \sec\left(\theta + \frac{2\pi}{3}\right) & \mathrm{cosec}\left(\theta + \frac{2\pi}{3}\right) & 1 \\ \sec\left(\theta + \frac{4\pi}{3}\right) & \mathrm{cosec}\left(\theta + \frac{4\pi}{3}\right) & 1 \end{vmatrix}$

Multiplying R_1, R_2 and R_3 by $\sin\theta\cos\theta$, $\sin\left(\theta + \frac{2\pi}{3}\right)\cos\left(\theta + \frac{2\pi}{3}\right)$, and $\sin\left(\theta + \frac{4\pi}{3}\right)\cos\left(\theta + \frac{4\pi}{3}\right)$ respectively, we get

$$\Delta = \frac{1}{k}\begin{vmatrix} \sin\theta & \cos\theta & \sin 2\theta \\ \sin\left(\theta + \frac{2\pi}{3}\right) & \cos\left(\theta + \frac{2\pi}{3}\right) & \sin\left(2\theta + \frac{4\pi}{3}\right) \\ \sin\left(\theta - \frac{2\pi}{3}\right) & \cos\left(\theta - \frac{2\pi}{3}\right) & \sin\left(2\theta - \frac{4\pi}{3}\right) \end{vmatrix}$$

where

$$k = \sin\theta\cos\theta\sin\left(\theta + \frac{2\pi}{3}\right)\cos\left(\theta + \frac{2\pi}{3}\right)\sin\left(\theta + \frac{4\pi}{3}\right)\cos\left(\theta + \frac{4\pi}{3}\right)$$

Operating $R_2 \to R_2 + R_3$

$$\Delta = \frac{1}{k}\begin{vmatrix} \sin\theta & \cos\theta & \sin 2\theta \\ 2\sin\theta\cdot\cos\frac{2\pi}{3} & 2\cos\theta\cdot\cos\frac{2\pi}{3} & 2\sin 2\theta\cdot\cos\frac{4\pi}{3} \\ \sin\left(\theta - \frac{2\pi}{3}\right) & \cos\left(\theta - \frac{2\pi}{3}\right) & \sin\left(2\theta - \frac{4\pi}{3}\right) \end{vmatrix}$$

$$= \frac{1}{k}\begin{vmatrix} \sin\theta & \cos\theta & \sin 2\theta \\ -\sin\theta & -\cos\theta & -\sin 2\theta \\ \sin\left(\theta - \frac{2\pi}{3}\right) & \cos\left(\theta - \frac{2\pi}{3}\right) & \sin\left(2\theta - \frac{4\pi}{3}\right) \end{vmatrix}$$

Hence, $\Delta = 0$.

6. Let the coordinates of P be $(a\cos\theta, b\sin\theta)$. Then the coordinates of Q are $(a\cos\theta, a\sin\theta)$.

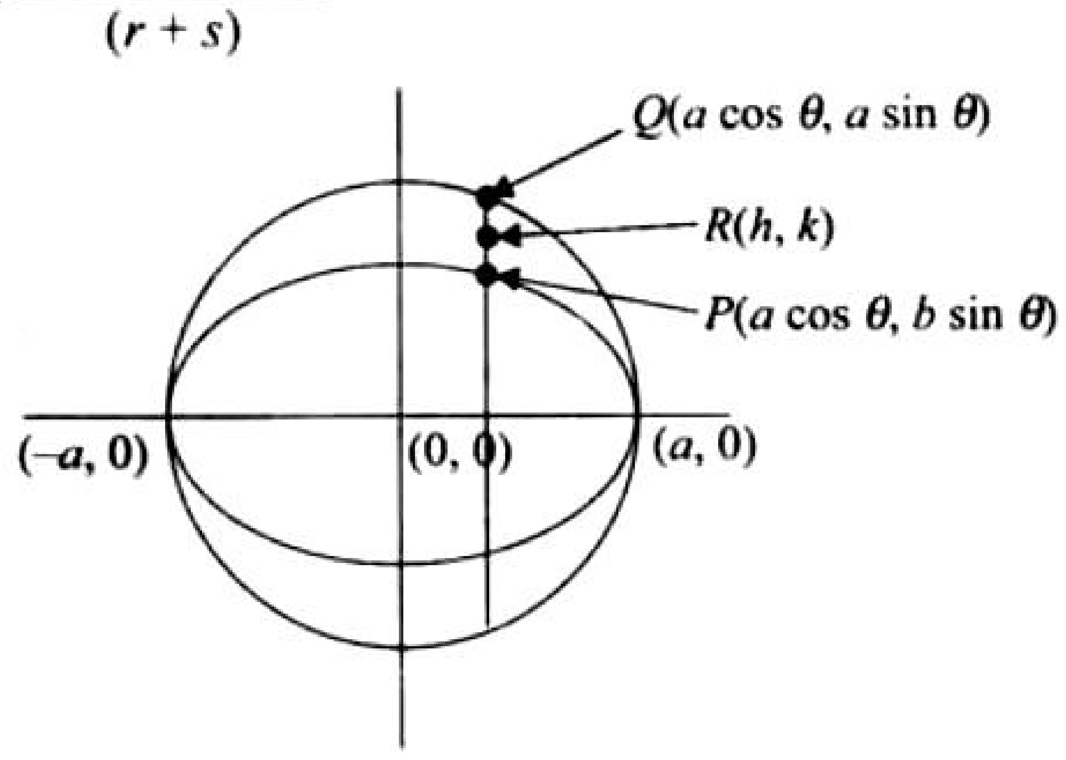
Let $R(h, k)$ divides PQ in the ratio $r : s$. Then,

$$h = \frac{s(a\cos\theta) + r(a\cos\theta)}{(r+s)} = a\cos\theta$$

or $\cos\theta = \frac{h}{a}$

$$k = \frac{s\,(b\sin\theta) + r\,(a\sin\theta)}{(r+s)}$$

$$= \frac{\sin\theta\,(bs + ar)}{(r+s)}$$



or $\sin\theta = \frac{k(r+s)}{(bs + ar)}$

We know that $\cos^2\theta + \sin^2\theta = 1$. Therefore,

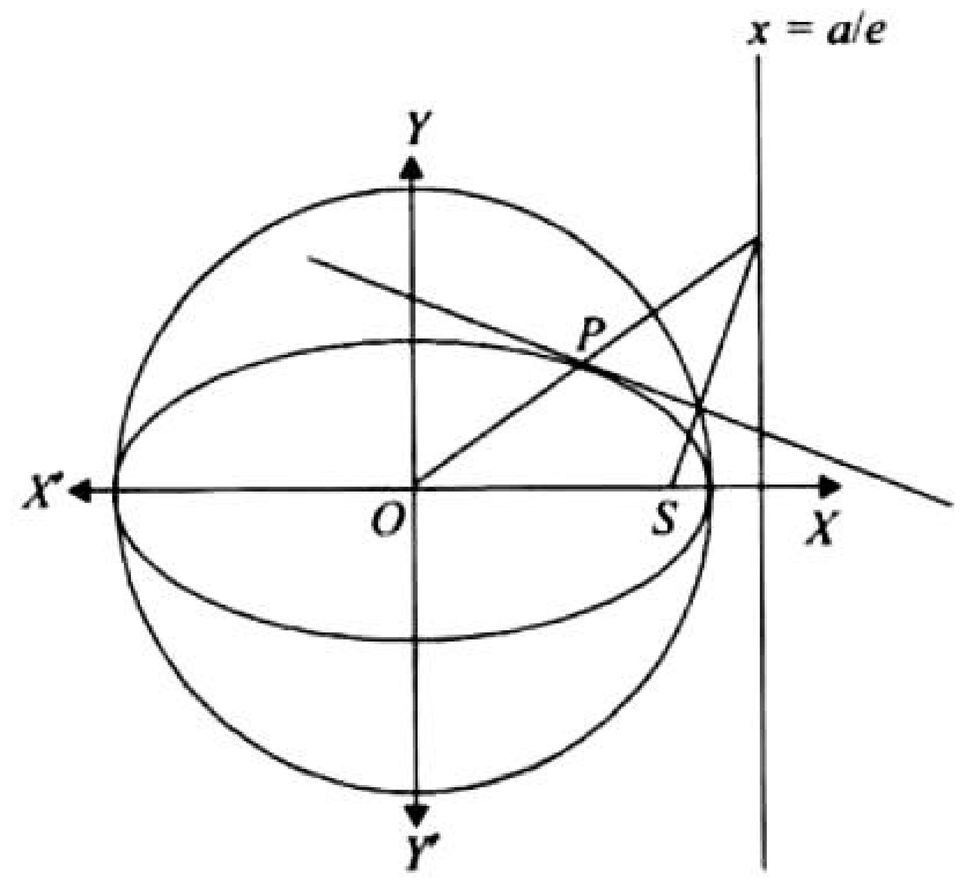
$$\frac{h^2}{a^2} + \frac{k^2(r+s)^2}{(bs+ar)^2} = 1$$

Hence, the locus of R is

$$\frac{x^2}{a^2} + \frac{y^2(r+s)^2}{(bs+ar)^2} = 1$$

which is an ellipse.

7.



Let the ellipse be

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

and O be the center.

The tangent at $P(x_1, y_1)$ is

$$\frac{xx_1}{a^2} + \frac{yy_1}{b^2} - 1 = 0$$

whose slope is $-b^2x_1/a^2y_1$.

The focus of the ellipse is $S(ae, 0)$.

The equation of the line through $S(ae, 0)$ perpendicular to the tangent at P is

$$y = \frac{a^2y_1}{b^2x_1}(x - ae) \qquad \text{(i)}$$

The equation of OP is

$$y = \frac{y_1}{x_1}x \qquad \text{(ii)}$$

Solving (i) and (ii), we get

$$\frac{y_1}{x_1}x = \frac{a^2y_1}{b^2x_1}(x - ae)$$

or $x(a^2 - b^2) = a^3e$

or $x\,a^2e^2 = a^3e$

or $x = \frac{a}{e}$

This is the corresponding directrix.

8. Any tangent on the ellipse $\frac{x^2}{25} + \frac{y^2}{4} = 1$ is

$y = mx \pm \sqrt{25m^2 + 4}$

If this is also the tangent on the circle, $x^2 + y^2 = 16$ then distance of center of circle from tangent is equal to radius

$$\therefore \left|\frac{0 - m \times 0 \pm \sqrt{25m^2+4}}{\sqrt{1+m^2}}\right| = 4$$

or $m = \pm\frac{2}{\sqrt{3}}$

Since the common tangent is in the first quadrant,

$m = -\frac{2}{\sqrt{3}}$

Hence, the common tangent in the first quadrant is given by

$$y = \frac{-2}{\sqrt{3}}x + \sqrt{\frac{112}{3}}$$

$$\sqrt{3}\,y + 2x = 4\sqrt{7} \qquad \text{(i)}$$

The points of intersection of this tangent with the x- and the y-axis are $(2\sqrt{7}, 0)$ and $(0, 4\sqrt{7}/\sqrt{3})$, respectively.

Therefore, the length of intercept is

$$\sqrt{(2\sqrt{7} - 0)^2 + \left(0 - \frac{4\sqrt{7}}{\sqrt{3}}\right)^2} = \frac{14}{\sqrt{3}}$$